The Irreducible Uncertainty of Ranking and Ordering

Constraint ranking in Optimality Theory (OT) is a type of ordering, and ordering has been studied in many domains. One challenging aspect of ranking constraints to fit data is the fact that there are often many rankings consistent with a given observation and, in some cases, these must be described with disjunction. Consider for example the abstract candidate comparison illustrated in (1).

(1)		*A*	*B*	*C*
	Cand 1	0	0	1
	Cand 2	1	1	0

Cand 1 is optimal if constraints *A* ***or*** *B outrank* constraint *C*. The ambiguity encoded by this disjunctive ranking statement makes it fundamentally more complex than its conjunctive converse, necessary for Cand 2 to be optimal: *C outranks A* ***and*** *B*. Conjunctive statements like this one can be combined to describe a partial order and, equivalently, they can be represented with Hasse diagrams. The same does not hold for statements with disjunction; they cannot be represented with Hasse diagrams and they do not correspond to partial orders. Merchant & Riggle (2016) use cases like this to prove that the mathematical structure needed to describe disjunctions of rankings in OT is that of an *antimatroid*, a more general order-theoretic class than partial orders which allows disjunction. **In this work, we provide a parallel proof that disjunctions arise in the description of rule orderings and thus that the mathematical structure needed to describe rule orderings is also that of an antimatroid.**

Despite the tradition of representing rule orderings as partial orders (which can in turn be represented with Hasse diagrams), the same kinds of disjunction as that sketched in (1) can arise regarding which rule orderings generate a given body of data. For example, consider the following (constructed) pattern: a language with fixed initial stress, followed by the rules in (2).

(2)	*A*	V: → 'V:	(long vowels are stressed)
	B	'V → 'V:	(stressed vowels are long)
	C	'a → 'e **or** a: → e:	(stressed **or** long low vowels are raised)

Because the combined action of rules *A* and *B* makes it such that stress and length always coincide, whether stress or length is responsible for the low vowel raising accomplished by rule *C* is technically indeterminate, as indicated above. If all three rules are surface true, and all stressed long vowels are indeed raised, this result can be achieved if rules *A* ***or*** *B precede* rule *C*. This is precisely the kind of disjunctive ordering statement that cannot be represented by a partial ordering and thus the mathematical structure needed to describe rule orderings consistent with observed data is, as with constraint ranking, that of an antimatroid.

Having shown that ranking and ordering admit the same type of ambiguity with respect to the set of grammars that are consistent with a body of observed data, one obvious ramification pertains to the way that generalizations to hitherto unseen data can be made in the face of the ambiguity inherent in disjunctively underdetermined rankings/orderings. Specifically, different resolution strategies for the ambiguity created by the disjunction can make different predictions about how grammars generalize to unseen data. In the scenario in (2), for example, the ambiguity surrounding the formulation of rule *C* can be resolved in either of the following two ways.

(3)	*A* < *C* < *B* ; *C* = 'a → 'e	#'a and 'a: raised by *C* after stressing of (non-initial) long vowels by *A*
	B < *C* < *A* ; *C* = a: → e:	#a: and a: raised by *C* after lengthening of (initial) stressed vowels by *B*

A parallel scenario can be constructed for constraint ranking. Observing that final codas are preserved, a learner can either generalize that MAX-C >> NOCODA or that ANCHOR-R >> NOCODA (see also Hayes 2004 and Prince & Tesar 2004). Resolving in favor of the former ranking leads to preservation of medial codas, while resolving in favor of the latter ranking leads to their deletion.

(4)	*A*	MAX-C	(consonants are not deleted)	/CVC/	*A*	*B*	*C*	/CVCCV/	*A*	*B*	*C*
	B	ANCHOR-R	(final segments are preserved finally)	.CVC.	0	0	1	.CVC.CV.	0	0	1
	C	NOCODA	(codas are marked)	.CV.	1	1	0	.CV.CV.	1	0	0

(5)	*A* >> *C* >> *B* ; /CVCCV/ → .CVC.CV.	medial codas preserved (*C* = NOCODA is violated everywhere)
	B >> *C* >> *A* ; /CVCCV/ → .CV.CV.	medial codas deleted (*C* = NOCODA is only violated finally)

Antimatroids are required for the description of many ordered systems such as consumer preferences, education prerequisites, and task queue scheduling (Korte & Lováscz 1984). Merchant & Riggle (2016) showed that antimatroids are needed to describe constraint rankings consistent with data, and we have demonstrated that the same is true for rule orderings. These results are not connected to any particular set of constraints or set of rules. Instead, they are a fundamental property of both constraint ranking and rule ordering and thereby reveal a deep similarity between the two regarding the kinds of uncertainty in fitting grammars to observations. To wit, for any given body of data the consistent rankings/orderings can be underdetermined in a disjunctive way — *A* **or** *B* outranks/precedes *C*.

When faced with ambiguous rankings/orderings (i.e., antimatroids), how do learners generalize to new forms that require a resolution of the ambiguity? This question has been examined for orderings and rankings by Durvasula & Liter (2020), Hayes (2004) and Prince & Tesar (2004), among others. What we have shown in this work is that, regardless of whether grammars are ranked constraints or ordered rules, the same questions will arise because we see the same irreducible uncertainty in ranking and ordering.

References cited: K. Durvasula, A. Liter. 2020. There is a simplicity bias when generalizing from ambiguous data. *Phonology* 37. • B. Hayes. 2004. Phonological acquisition in Optimality Theory: the early stages. In *Fixing Priorities: Constraints in Phonological Acquisition*. CUP. • B. Korte, L. Lovász. 1984. Greedoids, a structural framework for the greedy algorithm. In *Progress in combinatorial optimization, The Silver Jubilee Conference on Combinatorics*. Academic Press. • N. Merchant, J. Riggle. 2016. OT grammars, beyond partial orders: ERC sets and antimatroids. *NLLT* 34. • A. Prince. 2002. Entailed ranking arguments. ROA-500. • A. Prince, B. Tesar. 2004. Learning phonotactic distributions. In *Fixing Priorities: Constraints in Phonological Acquisition*. CUP.